

## SOME REMARKS ON CHAIN PROLONGATIONS IN DYNAMICAL SYSTEMS

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ABSTRACT. In this article, we discuss the notions of chain prolongation functions on locally compact spaces and get some results for the concepts. We show that chain prolongation function is a cluster map.

### 1. Introduction

Poincaré firstly introduced the concept of prolongations in a special sense. The notions of prolongations play an important role in studying dynamical systems [2, 3, 14]. In [15], Ura showed that the concept was closely related to the one of stabilities in the sense of Liapunov. Using the notion of prolongations, Auslander and Seibert defined several stabilities in [2]. Especially, they focused the notion of absolute stability of a compact subset in a locally compact metric space in terms of the concept of prolongation. Also, it is proved that the definitions were characterized by Liapunov functions and the presence of a fundamental system of absolutely stable neighborhoods in [2]. In recent year, in [14], Souza and Tozatti introduced the notions of prolongations and prolongational limit sets on control systems and generalized several important concepts and

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results of recursiveness and dispersiveness from Bhatia and Szegö (See [4]).

The concept of a pseudo-orbits(chains) was firstly used by Bowen [6] and Conley [8]. The notion is a very useful tool to understand important theories in the several fields of Mathematics and generates many results about the induced concepts, for example, chain transitive, chain recurrence, shadowing property and so on. See [1, 5, 10, 11, 12, 13, 16].

C. Ding introduced chain prolongation, with which he defined the concept of chain stability which is induced by the different notions of chain and stability and he obtained many interesting results for the concept in [9]. In particular, he proved that chain prolongation function is a cluster mapping but unfortunately we find some points in the proofs unclear.

In this article, we mainly discuss the properties of chain prolongation mappings. Also, we remove such unclear points in the proofs and so clarify them. More precisely, we get alternative proofs for the results related with chain prolongation functions and cluster maps.

## 2. Chain prolongation and stability

Let  $(X, d)$  be a locally compact metric space and  $\pi$  a flow, that is,  $\pi : X \times \mathbb{R} \rightarrow X$  is a continuous map such that  $\pi(x, 0) = x$  and  $\pi(\pi(x, t), s) = \pi(x, t + s)$  for  $x \in X$  and  $t, s \in \mathbb{R}$ . For the convenience, we briefly write  $x \cdot t = \pi(x, t)$ . For any  $x \in X$ , the *orbit* of  $x$  is defined by  $O(x) = \{x \cdot t \mid t \in \mathbb{R}\}$ . A subset  $Y$  of  $X$  is called *positively invariant* (*invariant*) under  $\pi$  if  $Y \cdot \mathbb{R}^+ = Y$  ( $Y \cdot \mathbb{R} = Y$ ). For a point  $x$  of  $X$ , the *limit set* of  $x$  is defined by

$$\Lambda^+(x) := \bigcap_{t \geq 0} \overline{x \cdot [t, \infty)}.$$

The limit set of  $x$  has a major role in Conley's theory, and for its basic properties we refer to [8, 9]. For  $x \in X$ , we also call the *first prolongational limit set* and *first prolongational set* of  $x$  as defined, respectively, by

$$J^+(x) := \bigcap_{U \in N(x), t \geq 0} \overline{U \cdot [t, \infty)},$$

$$D^+(x) := \bigcap_{U \in N(x)} \overline{U \cdot \mathbb{R}^+},$$

where  $N(x)$  is the set of all neighborhoods of  $x$ .

From the above definitions, we immediately obtain the following equivalences.

REMARK 2.1. For  $x \in X$ , the following equivalences hold.

- (1)  $y \in \Lambda^+(x)$  if and only if there is a sequence  $\{t_n\}$  in  $\mathbb{R}^+$  with  $t_n \rightarrow \infty$  such that  $x \cdot t_n \rightarrow y$ .
- (2)  $y \in J^+(x)$  if and only if there are a sequence  $\{x_n\}$  in  $X$  and a sequence  $\{t_n\}$  in  $\mathbb{R}^+$  such that  $x_n \rightarrow x$ ,  $t_n \rightarrow \infty$  and  $x_n \cdot t_n \rightarrow y$ .
- (3)  $y \in D^+(x)$  if and only if there are a sequence  $\{x_n\}$  in  $X$  and a sequence  $\{t_n\}$  in  $\mathbb{R}^+$  such that  $x_n \rightarrow x$  and  $x_n t_n \rightarrow y$ .

Let  $\Gamma : X \rightarrow 2^X$  be a set-valued function and  $A \subseteq X$ . Then we may canonically define the following equation,

$$\Gamma(A) = \cup_{x \in A} \Gamma(x).$$

We let the composition  $\Gamma^2 = \Gamma \circ \Gamma$  given by

$$\Gamma^2(x) = \Gamma(\Gamma(x)) = \cup_{y \in \Gamma(x)} \Gamma(y),$$

so we can define naturally the iteration  $\Gamma^n : X \rightarrow 2^X$  inductively by  $\Gamma^1(x) = \Gamma(x)$  and  $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$ . The trajectory for the function  $\Gamma$  is the union of the iteration  $\Gamma^n$ . For a family of functions  $\Gamma_i : X \rightarrow 2^X$  ( $i \in I$ ), we mean the map  $\cup_{i \in I} \Gamma_i : X \rightarrow 2^X$  defined by  $(\cup_{i \in I} \Gamma_i)(x) = \cup_{i \in I} (\Gamma_i(x))$ . So we can naturally define the functions  $D\Gamma$  and  $S\Gamma$  from  $X$  to  $2^X$  by

$$D\Gamma(x) := \cap_{U \in N(x)} \overline{\Gamma(U)} \quad \text{and} \quad S\Gamma(x) := \cup_{n=1}^{\infty} \Gamma^n(x).$$

LEMMA 2.2. Let  $\Gamma : X \rightarrow 2^X$  be a set-valued function on  $X$  and  $x \in X$ . Then  $D\Gamma(x)$  is the set of all points  $y \in X$  with the property that there exist sequences  $(x_n)$  and  $(y_n)$  in  $X$  with  $y_n \in \Gamma(x_n)$  such that  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ . Furthermore,  $S\Gamma(x)$  is the set of all points  $y \in X$  such that there is a finite subset  $\{x_1, \dots, x_k\}$  of  $X$  with the properties that  $x_1 = x$ ,  $x_k = y$  and  $x_{i+1} \in \Gamma(x_i)$ ,  $i = 1, \dots, k-1$ .

*Proof.* It is obvious from the definitions.  $\square$

The above new mappings have interesting properties, especially the mappings are idempotent. So it is clear that the iterations of the mappings are just the original mappings. See [7].

The map  $\Gamma : X \rightarrow 2^X$  is *transitive* provided  $S\Gamma = \Gamma$ . Note that  $\Gamma$  is transitive if  $\Gamma^2 = \Gamma$ . A mapping  $\Gamma$  is a *cluster mapping* if  $D\Gamma = \Gamma$ .

We recall the notions of chains and  $\Omega$ -limit sets and refer the reader to [8] for details. Let  $x, y$  be elements of  $X$  and  $\epsilon, t$  positive real numbers. An  $(\epsilon, t)$ -*chain* from  $x$  to  $y$  means a pair of finite sequences

$x = x_1, x_2, \dots, x_n, x_{n+1} = y$  in  $X$  and  $t_1, t_2, \dots, t_n$  in  $\mathbb{R}^+$  such that  $t_i \geq t$  and  $d(x_i \cdot t_i, x_{i+1}) \leq \epsilon$  for all  $i = 1, 2, \dots, n$ . Define a relation  $R$  in  $X \times X$  given by  $xRy$  means, for every  $\epsilon > 0$  and  $t > 0$ , there exists  $(\epsilon, t)$ -chain from  $x$  to  $y$ . We also denote  $(x, y) \in R$  by  $xRy$ .

For  $x \in X$ , we define the  $\Omega$ -limit set of  $x$  by  $\Omega(x) = \{y \in X : (x, y) \in R\}$ . We also canonically define a map  $\Omega : X \rightarrow 2^X$  given by  $x \mapsto \Omega(x)$ .

In [8], Conley studied the above notions in a compact metric space  $X$ . He proved that the relation  $R$  is a closed transitive relation on  $X$  and also showed that if  $(x, y) \in R$  and  $(s_1, s_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ , then  $(x \cdot s_1, y \cdot s_2) \in R$ .

Observe that  $\Omega(x)$  is a closed invariant subset of a compact metric space  $X$  and  $J^+(x) \subseteq \Omega(x)$ (see [8, p.36] and [9, p.2721]).

The map  $P : X \rightarrow 2^X$  defined by  $P(x) = x \cdot \mathbb{R}^+ \cup \Omega(x)$  is called a *chain prolongation*. For each  $x \in X$ , the image  $P(x) = x \cdot \mathbb{R}^+ \cup \Omega(x)$  of  $x$  for the mapping  $P$  is said to be the *chain prolongation set of  $x$* .

LEMMA 2.3. *If  $p_n \rightarrow p, q_n \rightarrow q$  and  $q_n \in \Omega(p_n)$ , then  $q \in \Omega(p)$ .*

*Proof.* From the continuity of the flow, for every  $\epsilon > 0, t > 0$ , there is a neighborhood  $V$  of  $p$  such that  $V \cdot t$  is contained in  $N_\epsilon(p \cdot t)$ . By assumption, we can choose a natural number  $n$  satisfying  $p_n \in V$  and  $q_n \in N_{\frac{\epsilon}{2}}(q)$ . So, since  $q_n \in \Omega(p_n)$ , there exists a  $(\frac{\epsilon}{2}, 2t)$ -chain  $\{p_n = x_1, \dots, x_m, x_{m+1} = q_n; t_1, \dots, t_m\}$  from  $p_n$  to  $q_n$ . Since  $p_n \in V$ , we gave that  $p_n \cdot t \in V \cdot t \subset N_\epsilon(p \cdot t)$  so  $d(p \cdot t, p_n \cdot t) < \epsilon$ . Furthermore,

$$d((p_n \cdot t)(t_1 - t), x_2) = d(p_n \cdot t_1, x_2) = d(x_1 \cdot t_1, x_2) \leq \frac{\epsilon}{2} < \epsilon,$$

$$d(x_m \cdot t_m, q) \leq d(x_m \cdot t_m, q_n) + d(q_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore,  $\{p, p_n \cdot t, x_2, \dots, x_m, q; t, t_1 - t, t_2, \dots, t_m\}$  is an  $(\epsilon, t)$ -chain from  $p$  to  $q$ . Hence, we obtain  $q \in \Omega(p)$  which completes the proof.  $\square$

REMARK 2.4. [9, p.2721] The followings are true.

- (1) The chain prolongation  $P(x)$  is positively invariant, and closed in  $X$ .
- (2)  $P(P(x)) = P(x)$  so the chain prolongation  $P$  is transitive.
- (3)  $C = \{(x, y) \in X \times X : x \in X, y \in P(x)\}$  is a closed and transitive relation.

In [9, p.2721], Ding defined the set  $P_t(x) := x \cdot [t, \infty) \cup \Omega(x)$  and then claimed that the set is identical to the following set

$$\{y \in X \mid \text{for each } \epsilon > 0, \text{ there is an } (\epsilon, t)\text{-chain from } x \text{ to } y\}.$$

Using this equality, Ding showed that the chain prolongation mapping  $P$  is a cluster map. But unfortunately, the above equality cannot be

guaranteed from the definition of  $P_t(x)$ . Then, we can consider the result without the equality and so we complete the above implication. In view of our proof, it is out of use the notion of  $P_t(x)$ .

**THEOREM 2.5.** *The chain prolongation function  $P$  is a cluster map, that is,*

$$(DP)(x) = P(x).$$

*Proof.* From the definition of  $DP(x)$ , we immediately obtain that  $P(x) \subseteq DP(x)$ .

Conversely, let  $y$  be an element of  $DP(x)$ . By Lemma 2.2, there are two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $y_n \in P(x_n)$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . From the definition of the chain prolongation  $P$ , we have two cases are depending upon whether the infinite subsequence  $\{y_{n_i}\}$  of the sequence  $\{y_n\}$  is contained in the set  $\cup_{i>0} x_{n_i} \cdot \mathbb{R}^+$  or  $\cup_{i>0} \Omega(x_{n_i})$ . Here, each  $y_{n_i}$  is an element of  $x_{n_i} \cdot \mathbb{R}^+$  or  $\Omega(x_{n_i})$ .

The first case is that a subsequence  $\{y_{n_i}\}$  is contained in  $\cup_{i>0} x_{n_i} \cdot \mathbb{R}^+$ . Thus we have that there exist positive real numbers  $t_i, (i \in \mathbb{N})$  such that  $y_{n_i} = x_{n_i} \cdot t_i$ . To show the inclusion, we begin by assuming that the sequence  $\{t_i\}$  is bounded. Since the sequence is bounded, there exists convergent subsequence of  $\{t_i\}$ . We can consider that the original sequence  $\{t_i\}$  is convergent to a positive real number  $t$ . We already knew that the subsequence  $y_{n_i}$  converges to  $y$ ,  $y_{n_i} = x_{n_i} \cdot t_i$  and  $x_{n_i} \cdot t_i \rightarrow x \cdot t$ . So we have that  $y = x \cdot t$ , that is,  $y$  is contained in  $x \cdot \mathbb{R}^+$ . Now, we consider  $\{t_i\}$  to be unbounded, so may assume that  $\{t_i\} \rightarrow \infty$ . Since  $x_{n_i} \cdot t_i = y_{n_i}$ ,  $x_{n_i} \rightarrow x$  and  $y_{n_i} \rightarrow y$ , it implies  $y \in J^+(x)$  by the remark 2.1. Since  $J^+(x) \subset \Omega(x)$ , we get that  $y \in \Omega(x)$ .

In the remaining case, the infinite subsequence  $\{y_{n_i}\}$  of  $\{y_n\}$  is contained in  $\cup_{i>0} \Omega(x_{n_i})$ . Since the subsequence  $\{x_{n_i}\}$  converges to  $x$  and the subsequence  $\{y_{n_i}\}$  also converges to  $y$ , by the lemma 2.3,  $y \in \Omega(x)$ . This completes the claim. Therefore, we prove that  $P(x) = DP(x)$ , that is,  $P$  is a cluster map.  $\square$

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